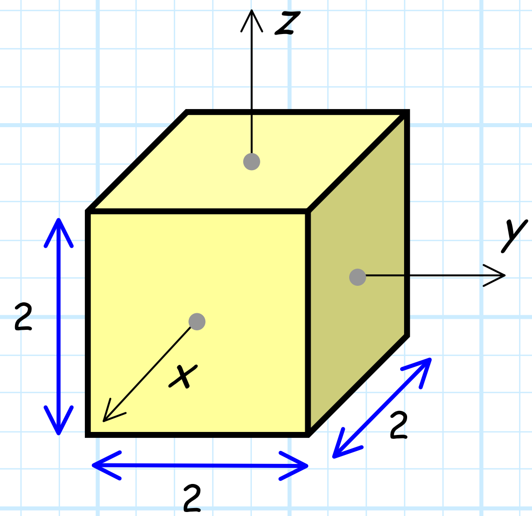
Integrals with Complex Surfaces

Similar to contours, we can form complex surfaces by combining any of the **seven** simple surfaces that can easily be formed with Cartesian, cylindrical or spherical coordinates. For example, we can define **6 planes** to form the surface of a **cube** centered at the origin:



The cube surface S is thus described as the sum of the **six** sides:

$$S = S_1 + S_2 + S_3 + S_4 + S_5 + S_6$$

Therefore, a surface integration over S can be evaluated as:

$$\iint_S \mathbf{A}(\bar{r}_s)\cdot\overline{ds} = \iint_{S_1} \mathbf{A}(\bar{r}_s)\cdot\overline{ds} + \iint_{S_2} \mathbf{A}(\bar{r}_s)\cdot\overline{ds} + \iint_{S_3} \mathbf{A}(\bar{r}_s)\cdot\overline{ds}$$
$$+ \iint_{S_4} \mathbf{A}(\bar{r}_s)\cdot\overline{ds} + \iint_{S_5} \mathbf{A}(\bar{r}_s)\cdot\overline{ds} + \iint_{S_6} \mathbf{A}(\bar{r}_s)\cdot\overline{ds}$$

This is a great example for considering the **direction** of differential surface vector $\overline{ds}$.

Recall there are **two** differential surface vectors that are orthogonal to every surface: the first is simply the **opposite** of the second.

For example, if we were performing a surface integration over the top surface of this cube (i.e., z=1 plane) , we would **typically** use $\overline{ds} = \overline{ds}_z = \hat{a}_z \, dx \, dy$.

However, we could **also** use the differential surface vector $\overline{ds} = -\overline{ds}_z = -\hat{a}_z \, dx \, dy$!

Q: *How would the results of the two integrations* ***differ?***

A: By a factor of **−1** !!

We find that a surface integration using $\overline{ds}$ is related to the surface integration using $-\overline{ds}$ as:

$$\iint_S \mathbf{A}(\overline{r}_s) \cdot (-\overline{ds}) = -\iint_S \mathbf{A}(\overline{r}_s) \cdot \overline{ds}$$

The surface of a cube is an example of a **closed surface.** A closed surface is a surface that **completely surrounds** some volume. You cannot get from **one side** of a closed surface to the **other** side without **passing through** the surface.

In other words, if your **beverage** is surrounded by a closed surface, better go get your **can opener**!

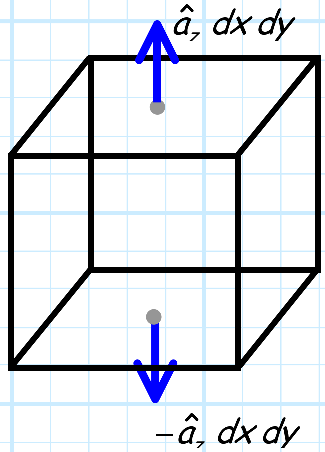
In electromagnetics, we **often** define $\overline{ds}$ as the direction **pointing outward** from a **closed** surface.

So, for example, the differential surface vector for the **top** surface (z=1) would be:

$$\overline{ds} = \overline{ds_z} = \hat{a}_z \, dx\, dy ,$$

while on the **bottom** (z=-1) we would use :

$$\overline{ds} = -\overline{ds_z} = -\hat{a}_z \, dx\, dy$$



Similarly, we would use differential line vectors of **opposite** directions for each of the pair of side surfaces (left and right), as well as for the front and back surfaces.